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Mirror symmetry and a G_2 flop

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Abstract: By applying mirror symmetry to D-branes in a Calabi-Yau geometry we shed light on a G_2 flop in M-theory relevant for large N dualities in $\mathcal{N}=1$ supersymmetric gauge theories. Furthermore, we derive superpotential for M-theory on corresponding G_2 manifolds for all A-D-E cases. This provides an effective method for geometric engineering of $\mathcal{N}=1$ gauge theories for which mirror symmetry gives exact information about vacuum geometry. We also find a number of interesting dual descriptions.

Keywords: M-Theory, String Duality.

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Contents

1. Introduction

It was proposed in [1] that large-N U(N) Chern-Simons gauge theory is dual to closed topological strings on the resolved conifold geometry.[1] There is now a large body of evidence supporting this conjecture including highly non-trivial exact computations to all orders in N [3]–[6].[2] On the other hand, this duality was embedded in type-IIA superstrings [8] where it was interpreted as a geometric transition starting with N D6 branes wrapped over S^3 of the conifold geometry, which gives an $\mathcal{N}=1$ U(N) gauge theory in $d=4$, and ending on the resolved conifold geometry where the branes have disappeared and been replaced by flux. In its mirror type-IIB formulation this is closely related to the dualities considered in [9, 10]. Extensions of this duality to other geometric transitions has been considered in [11] (see also [12, 13]).

On the other hand the lift of the duality of [8] to M-theory was considered in [14, 15] where both sides of the duality involve a smooth G_2 holonomy manifold with some quotient action. Moreover it was argued in [15] that the two geometries are quantum mechanically connected in a smooth way without any singularities, where $M2$ brane instantons would play a key role. The aim of this paper is to shed further light on this transition. In particular by employing a chain of dualities this transition gets related to type-IIA string

[1]In the original paper the gauge group was taken to be SU(N). Evidence has emerged [2] that this duality is more natural for the U(N) gauge group.

[2]This duality was also extended to the SO(N) and $SP(N)$ cases in [7].

theory in the presence of certain branes. In this context mirror symmetry can be employed, as discussed in [16, 17], to obtain exact quantum information for the theory by relating it to a classical type-IIB geometry. Not only this sheds light on the M-theory flop involving a G_2 holonomy manifold, but it can also be viewed as a program to geometrically engineer a large class of $\mathcal{N}=1$ gauge theories in the context of Calabi-Yau manifolds with branes.

The organization of this paper is as follows: In section 2 we recall the construction of special lagrangian A-branes in $\mathbb{C}^3$. In section 3 we review the lift of this geometry to M-theory [17]. In section 4 we discuss the application of mirror symmetry to this geometry and rederive the basic equation relating the size of the resolved $\mathbf{P}^1$ and the volume of S^3 [8]. In section 5 we consider generalizations to G_2 manifolds with A-D-E quotiont singularities. In section 6 we discuss some IR choices for the geometry that do affect the physical superpotential (in the dual Chern-Simons formulation this is related to the UV framing choice of the knot), and show that this ambiguity corresponds to the integer in the choice of the triple self-intersection of the Kahler class in the resolved conifold geometry [8]. In section 7 we discuss a number of dual descriptions, including 3 type-IIB descriptions, 3 type IIA descriptions and 2 M-theory descriptions. In one of the dual M-theory descriptions the whole geometry is replaced by a single M5 brane.

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2. Special lagrangian D6 branes in $\mathbb{C}^3$

In [16, 17] type-IIA string theory was studied where the background involves D6 branes wrapped on certain class of special lagrangian submanifolds L embedded in the local A-model geometry. The families of lagrangians are similar to the original examples of Harvey and Lawson [18], that were recently thoroughly studied by Joyce [19]. In fact, in the limit where all sizes of the cycles of the Calabi-Yau manifold are taken to infinity and the geometry is locally $\mathbb{C}^3$, the lagrangians L are *exactly* those of [18, 19]. In this section we review this construction. In the next section we recall [17] how this geometry in the limit of strong type-IIA string coupling is given by M-theory in a G_2 holonomy geometry which is topologically given by $\mathbf{S}^3 \times \mathbb{R}^4$.

Consider $\mathbb{C}^3$ with coordinates x_1, x_2, x_3 and a flat metric. Following [16] consider the moment map $x_i \to |x_i|^2$, the image of which is $\mathbb{R}^3_+$. The fiber of this map is the torus of phases $x_i = |x_i| e^{i\theta_i}$, generically a T^3. We consider special lagrangian submanifold L of $\mathbb{C}^3$ of topology $\mathbf{S}^1 \times \mathbb{C}$ which is given by the following set of equations

$$\begin{aligned} |x_1|^2 - |x_2|^2 &= c_1 \\ |x_3|^2 - |x_2|^2 &= c_2 \end{aligned}$$

and $\sum_i \theta_i = 0$.

The image of L under the moment map is a line (figure 1) and the moduli c_i must be such that the line intersects the boundary of $\mathbb{R}^3_+$ along its one-dimensional edges, because L would not have been a complete manifold otherwise. The configuration space of D6 brane wrapping L is complexified by the Wilson-line around the S^1. There the S^1 that L wraps degenerates at a point, so the classical configuration space is that of three copies of

$\mathbb{C}$ meeting at a point. These phases are given by the conditions:

$$\begin{aligned} \text{Phase I}: &\quad c_2 = 0\,, c_1 > 0 \\ \text{Phase II}: &\quad c_1 = 0\,, c_2 > 0 \\ \text{Phase III}: &\quad c_1 = c_2 < 0\,. \end{aligned} \tag{2.1}$$

It is also convenient to rewrite the equations defining the locus of the D6 brane as follows: Let us restrict attention to Phase I. Then we can write the D6 brane locus as

$$|x_1|^2 - |x_3|^2 = c_1\,, \qquad x_2 = \overline{x}_3 e^{-i\theta_1}\,. \tag{2.2}$$

Note that the D6 brane has worldvolume with the topology of $\mathbb{C} \times \mathbf{S}^1$ where $\mathbb{C}$ can be identified with x_3.

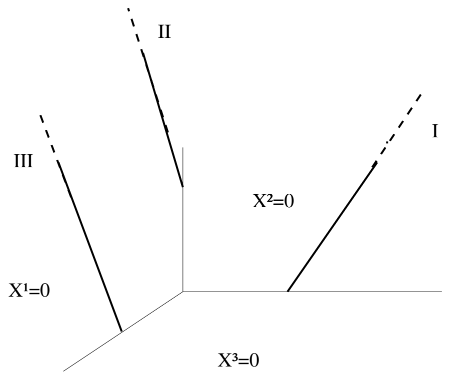

Figure 1: The D6 brane in the three phases is depicted here.

There are instanton corrections to the classical geometry which come from worldsheets which are disks ending on the $D6$ brane. For example, for phase I, for a D6 brane at $c_2 = 0$, and c_1 a positive constant, the primitive disc D is the holomorphic curve $x_2 = 0 = x_3$ bounded by $|x_1|^2 \leq c_1$. The disc partition function $F_{0,1}$ of the A-model topological string computes the exact superpotential of the type-IIA string theory in this background. The instanton corrections can be summed up exactly using mirror symmetry as discussed in [16, 17], as will be reviewed below.

3. Lift to M theory

The D6 branes of IIA string theory are Kaluza-Klein monopoles of M theory, so that the configuration we discussed above lifts to a purely geometric background of M-theory. In the absence of $D6$ branes type-IIA string theory on X lifts to M theory on $M^7 \sim X \times S^1$. Adding $D6$ branes compactified on L results in a manifold which is still locally $X \times S^1$ but where the S^1 degenerates over L. This makes M^7 into a seven-manifold which is not a product manifold and since the theory has $\mathcal{N} = 1$ supersymmetry M^7 must have G_2 holonomy [14, 15, 20, 21].

Since the $D6$ branes are geometrized at strong coupling the disc D discussed above must map to a complete manifold as well. The lift of the disc is locally $D \times S^1$ but the S^1 degenerates over the boundaries of the disc, which leads to a manifold of S^3 topology. We can model this $S^3 \subset M^7$ by writing $|x_1|^2 + |x_m|^2 = c_1$ since at generic value of $|x_1|$ this equation determines x_m up to a phase which is an S^1, but at $|x_1|^2 = c_1$, the size of the S^1 vanishes. The volume of S^3 can be identified with c_1.

It is clear that there is no other non-trivial cycles in M^7 and topologically it can be identified with $\mathbf{S}^3 \times \mathbb{R}^4$. More globally we can view x_m as a complex number whose norm $|x_m|$ varies over $\mathbb{C}^3$ where we identify its phase as the M-theory circle. Moreover

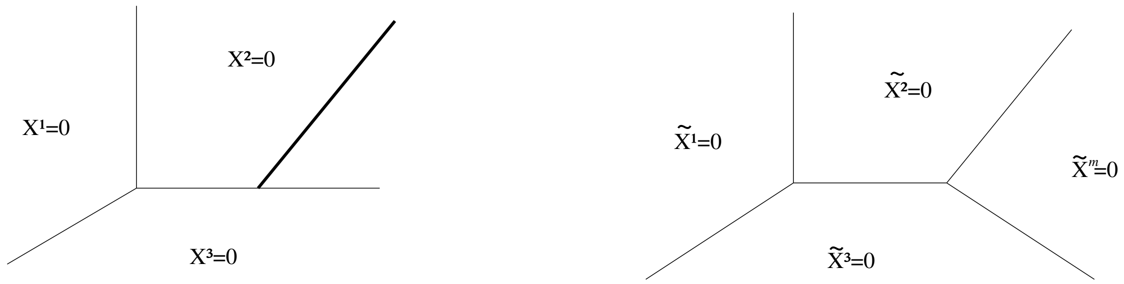

Figure 2: The enlarged moment map from $M^7 \to R^3$ (with a suitable choice of coordinates) treats all variables in a more symmetrical way.

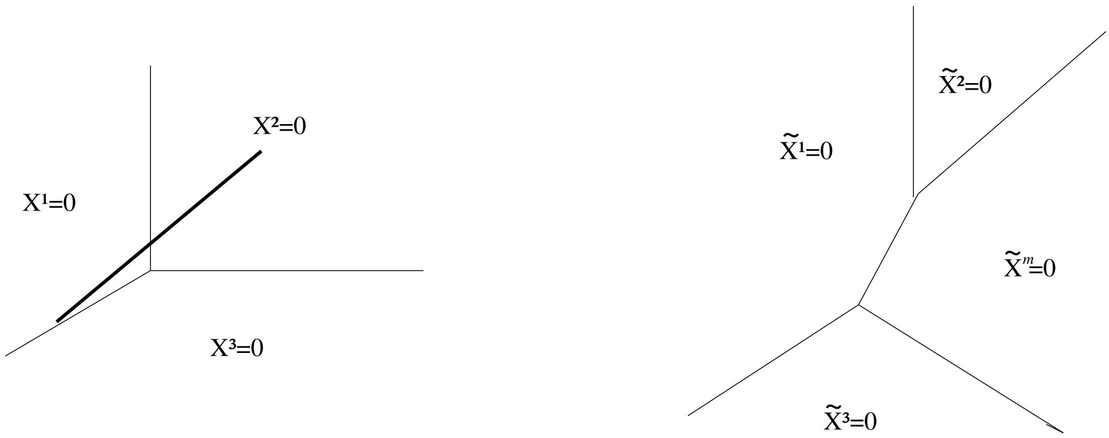

Figure 3: The second phase and its toric base.

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x_m vanishes over the special lagrangian submanifold L. There is also a natural M-theory projection of the enlarged moment map from M^7 to a subspace of R^3 given by a suitable choice of coordinates $\tilde{x}_i$ and projections $\tilde{x}_i \to |\tilde{x}_i|^2$, for $i = 1, 2, 3, m$ (see figure 2).

As discussed in [15] M theory has another phase where the volume of S^3 becomes negative. Let us define $r = c_1 - c_2$. In the first phase, where $c_2 = 0$, r is the volume of the S^3. If we take r to negative values $r \to -r$, we can identify this as the Phase II of the brane in the type-IIA geometry. This is where the role of c_1 and c_2 are exchanged. Now $c_1 = 0$ and $c_2 > 0$ denotes the volume of the new S^3 (see figure 3).

There is a third phase where $c_1 = c_2$ on the type IIA D6 brane, i.e. Phase III. This corresponds to $r = c_1 - c_2 = 0$, and it is difficult to see this from M-theory perspective. However the type-IIA perspective suggests again that there is an S^3 that can be blown up. The existence of the three phase structure in this context has been pointed out by Atiyah and Witten [22].

In [14, 15] the M theory on this same G_2 manifold was considered. The aim there was to fully geometrize the duality of [8]. In this context a different circle action was identified with the 11-th circle where in phase I leads to the S^3 being the fixed point locus, which in type-IIA is interpreted as a single D6 brane wrapping S^3. In phase II this same circle action acts freely and the corresponding geometry in the type-IIA description is the resolved conifold geometry where S^3 of M-theory maps to the S^2 of type-IIA dictated by the Hopf-fibration. This induces one unit of RR 2-form flux through S^2. This transition in type-IIA string theory is the $N = 1$ case of the transitions considered in [8] as there is only a single $D6$ brane present. It was shown in [8] that the parameters of the two descriptions, i.e., the complexified gauge coupling Y of the D6 brane theory, $Y = Vol(S^3) + i \int_{S^3} C$, and the complexified size of the two sphere t are related by

$$1 - e^{-t} = e^{-Y}. \tag{3.1}$$

Since the two theories exist for any value of t and Y, this can be viewed as an equation for a Riemann surface Σ as a hypersurface in $C^* \times C^*$ (parameterized by t, Y). We will now show that this equation also predicts the existence of three classical phases.

This is an auxiliary Riemann surface in the present language, but notice the following. In the language of the first IIA/M theory duality presented above, both the volume of the S^3 which is $re(Y)$ and the size of the S^2, the $re(t)$ become *geometric* parameters. Namely in the extreme limits of the three phases we have discussed, we have

$$\begin{aligned} \text{Phase I}: &\qquad t = 0\,, Y \gg 0 \\ \text{Phase II}: &\qquad t \gg 0\,, Y = 0 \\ \text{Phase III}: &\qquad t = Y \ll 0\,. \end{aligned} \tag{3.2}$$

Notice that these lie, approximately, on the Riemann surface (3.1). While it is true that the classical configuration space of $D6$ branes on L resembles the geometry of the Riemann surface above only approximately, and in the degenerate limit, its quantum geometry is in fact *exactly* that of Σ. Furthermore, it appears as the *classical* geometry of the mirror type-IIB string theory, which we now turn to.

4. Mirror symmetry

Using the recent results on mirror symmetry [23, 24] one can gain further insight into these type-IIA geometries with branes. In particular the mirror B-model of the $\mathbb{C}^3$ is given by [17]

$$xz = e^{-u} + e^{-v} + 1\,.$$

Under mirror symmetry, the D6 brane wrapped on L maps to a D5 brane on a holomorphic curve. The curve is given by $x = 0$, a choice of a point on the Riemann surface

$$0 = e^{-u} + e^{-v} + 1\,, \tag{4.1}$$

and fills the z plane. The B model geometry receives no quantum corrections and the theory is exact at the classical level. In the classical regime of the A model, as explained in [16, 17], the A and the B model geometry are related by a simple map. In phase I and for very large c_1, the $re(u)$ can be taken to be the size of the primitive disc. The imaginary part of u comes from the Wilson line around the S^1 which is finite in this phase. The c_2 itself is the real part of v, which vanishes in this limit, but after the phase transition to II and for large values of c_2, it is v which is related to the disc size. More generally however, the configuration space is the smooth Riemann surface Σ. From the identification of u, v as the variables measuring the sizes of the discs in the two limits, it is clear that we have the map

$$Y = \hat{u} \qquad t = \hat{v}\,,$$

where $\hat{u} = u + i\pi$ and $\hat{v} = v + i\pi$ denote the flat coordinates [17] (in our discussion below we sometimes revert back to the u, v notation for the flat coordinate instead of $\hat{u}, \hat{v}$). Thus the two Riemann surfaces (3.1), (4.1) are in fact canonically identified! Furthermore, mirror symmetry allows us to directly compute the superpotential of the $\mathcal{N} = 1$ theory [16], which explains why these Riemann surfaces are the same. Namely, the equation solved in [8] to yield (3.1) came from minimizing the space-time superpotential W,

$$\frac{dW}{dt} = N\frac{\partial^2 F}{\partial t^2} - Y = 0 \tag{4.2}$$

where, in the case we are considering here $N = 1$, and where F is the prepotential of the $O(-1) + O(-1) \to \mathbf{P}^1$

$$F = p\frac{t^3}{6} + \sum_{n>0} \frac{e^{-nt}}{n^3}\,.$$

The integer p is given by the triple self-intersection of the Kahler class, but it is somewhat ambiguous because of non-compactness of the class and depends on the choices made at infinity. In [8] p was set to a particular value, but in principle it can be any integer. In this section we will set it to zero and return to the general case in section 6.

On the other hand as discussed in [16] in the B-model the superpotential of the D5 brane geometry is given by

$$\frac{dW}{dv} = u(v) - u_0$$

where $u(v)$ is found by the condition (4.1) and u_0 is an additional constant term which is not fixed by mirror symmetry and does not affect the open topological string amplitudes. Setting $dW/dv = 0$ gives $u = u_0$ and $v(u)$ and thus the Riemann surface parameterizes the space of solutions as we change the volume of the S^3 in the original formulation [8]. Not only the condition to get the minima agree, but as one can readily check, the superpotential W of [8] is the same as the one obtained here by the methods of [16, 17], namely

$$W = -\sum_{n>0} \frac{e^{-nt}}{n^2} - Yt = -\sum_{n>0} \frac{e^{-n\hat{v}}}{n^2} - \hat{u}_0\hat{v}\,. \tag{4.3}$$

Thus, we can view this as an alternate derivation of the superpotential using mirror symmetry (and a chain of dualities!).

5. Generalizations to A-D-E quotients

In this section we study M-theory on quotients $M^7 = S^3 \times R^4/\Gamma$, where Γ is a discrete A-D-E subgroup of SU(2). In one phase Γ fixes S^3 and this gives rise to the corresponding $\mathcal{N} = 1$ supersymmetric A-D-E gauge symmetry in four dimensions. In the phase where the S^3 is flopped (and has formally large negative volume with respect to the original S^3) the Γ acts freely on S^3 and gives the lens space S^3/Γ. We would like to generalize the map we found using mirror symmetry between the sizes of the S^3's to these cases as well. For A and D there already is a prediction based on [8, 7]. There is no formula known for the E-series. In this section we rederive the result for the A case using mirror symmetry techniques. For the D and E series, even though one can still use mirror symmetry in principle, we use a shortcut to obtain the map. The idea is to use the relation between the superpotential and the domain walls to obtain this result. In this context we will use the results in [25].

5.1 The A_{N-1} case

Start with N D6 branes wrapping S^3 as in [8]. The lift of this to M-theory [14, 15] is given by the Z_N quotient acting on M^7, which fixes the S^3 in phase I. This in particular means that we have in terms of IIA variables

$$(x_1, x_2, x_3) \to \left(x_1, \omega x_2, \omega^{-1} x_3\right)$$

where $\omega^N = 1$. This leads to an A_{N-1} singularity in the type IIA geometry. However, this is not the end of the story, as we also have a D6 brane which is the vanishing locus of the circle action. Note that this is given by the same lagrangian submanifold we started with except orbifolded with Z_N:

$$L = \mathbb{C} \times \mathbf{S}^1 \to \mathbb{C}/\mathbb{Z}_N \times S^1 \, .$$

We will not repeat the phase structure analysis discussed before, as it is already discussed in [15, 14] and limit ourselves here to noting how the mirror type-IIB geometry is modified.

To find the mirror of this geometry with the brane, we first have to find the mirror of the underlying space. This has been done (in more generality[3]) in [26, 27, 28] and leads to

$$xz = e^{-u} + P_N(e^{-v})$$

where P_N is a polynomial of order N in e^{-v}. Putting the extra D6 brane in the geometry freezes the polynomial P_N to a particular value of coefficients which we will now determine. The mirror D5 brane is at $x = 0$, filling the z-space. From the A-model side in the limit where the brane is in phase I and when the disc instanton action is very small, the classical picture is accurate. In this limit we have $u \to \infty$ and the brane is at the classical value $v \sim 0$. Moreover there should be a unique point for this choice, as there is a unique brane allowed in the A-model geometry (the $\mathbb{C}/\mathbb{Z}_N \times \mathbf{S}^1$). This means that P_N should have N degenerate roots at $v = 0$, so that (up to a choice of a constant that can be absorbed into redefinition of variables):

$$xz = e^{-u} - (1 - e^{-v})^N \, .$$

[3]This can be viewed as a degeneration of the A_{N-1} fibered geometry over $\mathbf{P}^1$, for which the mirror is given by $xz = e^{-u} + P_N(e^{-v}) + \Lambda e^u$, where we take $\Lambda \to 0$ and thereby taking the size of $\mathbf{P}^1$ to infinity.

This leads to the Riemann surface

$$x = 0\,, \qquad (1 - e^{-v})^N = e^{-u}$$

and, using the identification $u \leftrightarrow Y$ and $v \leftrightarrow t$, gives the equation

$$\left(1 - e^{-t}\right)^N = e^{-Y} \tag{5.1}$$

in precise agreement with the relation between t and Y obtained in [8]. Note that here there is more information than just the choice of the Riemann surface, which parameterizes the gauge theory moduli. In fact, as discussed in [8] the duality of [1] (combined with the results of [29]) identifies

$$t \sim \langle \mathrm{tr}\mathcal{W}^2 \rangle$$

where $\mathcal{W}$ denotes the chiral field whose bottom component denotes the gaugino field of the gauge theory. Here, we can *derive* this. To see this, note that on one hand, shifting of the theta angle $Y \to Y + 2\pi i$ across the domain wall is accompanied by the change of the superpotential by $\Delta W = \langle \mathrm{tr}\mathcal{W}^2 \rangle$, and on the other hand using $Y = \hat{u}$ and the superpotential $W = \int u dv$, we have $\Delta W = \hat{v}$ is the tension of the corresponding domain wall [17], which provides the claimed identification. Thus, by viewing $t = tr\mathcal{W}^2$, $Y = 1/g^2_{YM}$ as functions on the Riemann surface, defined up to shifts by $2\pi i$, they satisfy a single relation which is the definition of the Riemann surface. Thus the parametrization of the Riemann surface in terms of t, Y carries important additional physical information. Note that Y is *also* the tension of a domain wall [17] and the Riemann surface can be viewed as the relation between the BPS tensions of the two types of domain walls. These domain walls are realized by M5 branes wrapped over S^3's on the various phases. As shown in [3] the structure of the superpotential dictates the degeneracy of the domain walls. In particular for cases at hand, the prediction of [3] is

$$W = \sum_{k>0} \sum_{n=1}^{\infty} d_k \frac{e^{-nkv}}{n^2} \tag{5.2}$$

where d_k counts the number of primitive of domain walls in the class given by kv, i.e. where the BPS tension of the domain wall is kv. For the case at hand, as discussed in [16, 17] we have

$$\frac{dW}{dv} = u = -N\log(1 - e^{-v}) = \sum_n N \frac{e^{-nv}}{n}$$

which leads to the superpotential

$$W = -\sum_{n=1}^{\infty} N \frac{e^{-nv}}{n^2}\,. \tag{5.3}$$

Comparing to (5.2) we see that $|d_1| = N$ and $d_k = 0$ for all $k > 0$.

On the other hand, the structure of the domain walls for M-theory on G_2 holonomy manifolds of the form $S^3 \times R^4/\Gamma$ where Γ acts as an A-D-E subgroup of SU(2) on S^3 and leading to lens space, was studied in [25]. The counting of the domain walls there was carried out upon compactification of this theory to three dimensions, as the question of counting of the domain walls in *four* dimensions was found to be somewhat ambiguous, and to depend on boundary conditions on the domain wall. The answer in *three* dimensions was found to have a simple structure: Consider for each A-D-E group Γ the corresponding affine Dynkin diagram. Then there is a *primitive* domain wall associated to each node i of the affine Dynkin diagram, with BPS charge given by the Dynkin number a_i. Moreover they form bound states for all possible combinations of the primitive domain walls consistent with assigning fermionic statistics to the primitive ones. The structure of the primitive domain walls follows the same situation studied for the case of modding out conifold T^*S^3 by A-D-E subgroups acting on S^3 done in [30] where it was shown that the primitive bound states are in one to one correspondence with irreducible representations of A-D-E with charge given by the dimension of the irreducible representation. This in turn is in one to one correspondence, using the McKay correspondence, with the nodes of the affine Dynkin diagram where the dimension of the representation is mapped to the Dynkin index.

For the A_{N-1} case, the primitive domain walls exist only in degree one, and there is one of each for every node of the affine Dynkin diagram (as all the Dynkin numbers are 1), i.e., $|d_1| = 1$. This agrees with the formula we found above (5.3), and so it leads to the identification of the primitive domain walls in 3 dimensions with the domain wall count given by the superpotential in four dimensions.

For a general Γ, the identification of the primitive domain walls of the 4 dimensional theory with those of the theory in three dimensions, which correspond to nodes of the affine Dynkin diagram and whose BPS charge is given by the corresponding Dynkin index, together with the relation of the general structure of the superpotential (5.2) to the counting of the primitive domain walls, suggests the superpotential is

$$W = -\sum_{i=0}^{r}\sum_{n>0}\frac{e^{-na_i v}}{n^2} \tag{5.4}$$

where r denotes the rank of the corresponding A-D-E. This leads to

$$u = \frac{dW}{dv} = -\sum_i a_i \log(1 - e^{-a_i v})$$

which in turn leads to

$$\prod_{i=0}^{r}\left(1 - e^{-a_i v}\right)^{a_i} = e^{-u}\,.$$

Let us write this as

$$\prod_{i=0}^{r}(1 - x^{a_i})^{a_i} = y \tag{5.5}$$

where $x = e^{-v}$ and $y = e^{-u}$.

We have already discussed the A_{N-1} case. We now present further evidence for this in D and E cases.

5.2 The D and E cases

The case of D_N was studied in [7] using the large-N limit of SO($2N$) Chern-Simons theory, and it was shown that there are two sources for the superpotential, after making the transition to the blown up $\mathbf{P}^1$ in the type-IIA orientifold setup: The contribution of the genus zero closed Riemann surface with $(2N-4)$ units of RR flux to the superpotential gives

$$W_0 = -(2N-4)\sum_{n>0}\frac{e^{-2nv}}{n^2}$$

where $v = t/2$ to compare with the notation of [7]. Moreover the RP^2 diagram also contributes to the superpotential giving,

$$W_1 = -2\sum_{n>0}\frac{(1-(-1)^n)e^{-nv}}{n^2}$$

One can see that

$$dW = dW_0 + dW_1 = -(2N-4)\log(1-x^2) - 2\log\frac{(1-x)}{(1+x)}$$

which leads to

$$y = e^{-u} = e^{-dW} = \frac{(1-x^2)^{2N-4}(1-x)^2}{(1+x)^2} = (1-x^2)^{2N-6}(1-x)^4$$

which is in perfect agreement with (5.5), when one recalls that for D_N, we have four nodes with Dynkin number 1 and $N-3$ nodes of Dynkin number 2.

We can also provide further evidence for (5.5) by various other considerations: Note that when $y \gg 0$, which corresponds to the volume of the original S^3 to be negative and large, this equation reduces to

$$x^{\sum a_i^2} = y \rightarrow x^{|\Gamma|} = y$$

where we have used the fact that the sum of the squares of the dimension of irreducible representations add up to the order of the group $|\Gamma|$. This is consistent with the fact that $-v$ represents the volume of the flopped S^3 and on the flopped side Γ acts freely and so the corresponding volume (including the imaginary piece) is reduced by a factor of $|\Gamma|$. Another general check one can make is the following: When the volume of the original S^3 is large, i.e., when $u \gg 0$ which corresponds to $y \sim 0$ we see that $x \sim 1$, i.e., $v \sim 0$ gives a solution to (5.5) given by

$$\prod_{i=0}^{r}(a_i v)^{a_i} = e^{-u}$$

which leads to

$$v \sim e^{-u/c_2}$$

where $c_2 = \sum_{i=0}^{r} a_i$ is the dual coxeter number of the corresponding group. This is consistent with the fact that in this limit we do have a decoupled $\mathcal{N}=1$ gauge theory of the

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A-D-E type and the gaugino condensate, identified with v, is expected to be given exactly by the above formula.

Note however, that even in this limit there are more solutions to (5.5), namely by considering $x = e^{-v}$ near other a_i-th roots of unity. Note that these correspond to points where the gaugino condensate differs by a shift $\mathrm{tr}\mathcal{W}^2 \to \mathrm{tr}\mathcal{W}^2 + 2\pi n/a_i$. Restoring units of M_{planck}, this shows that in the decoupling limit where $M_{planck} \to \infty$ these vacua are infinitely far away and decoupled from the A-D-E gauge theory. Nevertheless it has been argued in [22] that for some of these vacua one can find alternative gauge theory description by considering some discrete fluxes turned on in the R^4/Γ, that were studied in [31], on the original side.

Even though we have derived the (5.5) in the A_{N-1} cases using mirror symmetry and argued for its general structure for all A-D-E, it would be interesting to derive it directly from mirror symmetry for the D and E cases as well.

6. Framing ambiguity

In [17] it was pointed out that the choice of the IR geometry does modify the quantum aspects of the theory in the presence of branes, and in particular it modifies the superpotential and the Riemann surface relevant for the mirror geometry. The ambiguity was reflected by an integer p, and for example for $\mathbb{C}^3$ the Riemann surface was modified to

$$e^{-\hat{u}+p\hat{v}} + e^{-\hat{v}} - 1 = 0\,.$$

One would naturally ask, given the equivalence of this Riemann surface to the equation found in [8] relating t to Y whether there is also an ambiguity in there. In fact as well known the triple intersection of the basic non-compact 4-cycle class is ambiguous and depends on what one fixes at infinity. This affects the prepotential by introducing the classical term

$$F = p\frac{t^3}{6} + O(e^{-t})$$

where the classical self-intersection is taken to be the integer p. If one extremizes the superpotential with this F by solving (4.2) one obtains

$$(1 - e^{-t}) = e^{-Y+pt}$$

exactly as expected from the above ambiguity[4] of the Riemann surface!

7. Gymnastics on the duality web

In this section we relate some of the considerations in this paper to other dual descriptions. All said, we will end up with three type-IIA descriptions, three type-IIB descriptions, and two M-theory descriptions (not counting the different phases in each case).

[4]The generalization of this in the A_{N-1} case is $(1 - e^{-t})^N = e^{-Y+Npt}$.

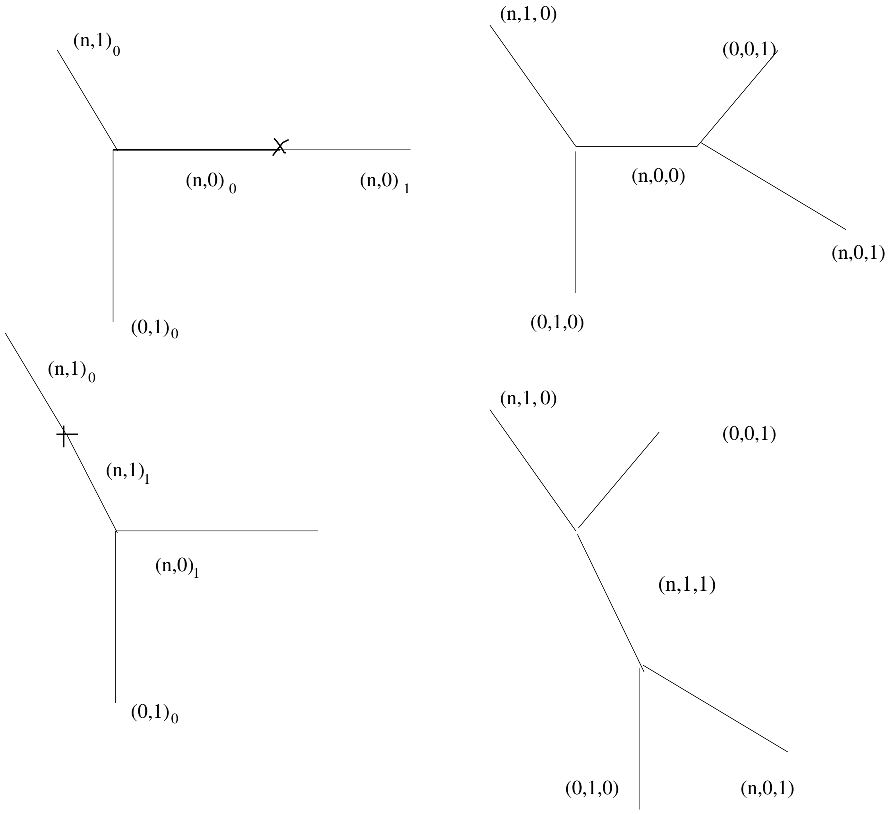

Figure 4: Flop transition as viewed from the perspective of (p, q) 5-branes of type-IIB.

Let us start with two of the type-IIA descriptions: Consider type IIA strings on the conifold geometry T^*S^3 with N D6 branes wrapping S^3. As we noted in section 5 this is dual to type-IIA on $\mathbb{C}^3/\mathbb{Z}_N$ with one non-compact D6 brane. Both of these lift to the same M-theory geometry involving a G_2 holonomy background with an S^3 at the fixed locus of a $\mathbb{Z}_N$ action, and we used this to argue their equivalence. Acting by mirror symmetry on the two type-IIA theories naturally leads to two type-IIB descriptions: The first one leads to N D5 branes wrapping S^2 and the second one gives one non-compact D5 brane in a Calabi-Yau geometry which involves a deformations of A_{N-1} geometry fibered over another space. As discussed in [17] there is also yet another dual type IIB description involving the web of (p, q) 5-branes in a background of ALF-like geometries. This can be obtained by starting from the M-theory geometry and identifying a T^2 in it (together with the equivalence of M-theory on T^2 with type-IIB on S^1). In this description the circle fibration of the ALF-like geometry is associated to a *third* circle action. This relates to the "extended" moment map we discussed in section 3. It is natural to use a notation which treats all three circle actions on the same footing, in which case the vanishing loci of the circle actions for the case of N branes are given by figure 4. Note that the disappearance of the U(N) symmetry in the flopped phase is clear, as depicted in figure'4; namely after the transition the intermediate brane appears with multiplicity one rather than multiplicity N.

Up to now, we have three type-IIB descriptions, two type-IIA descriptions and one M-theory description. Now we start with the type-IIB description in the form of a local CY geometry:

$$xz = F(u,v)$$

with a D5 brane wrapping z plane at $x = 0 = F(u,v)$. Let us T-dualize this only over the circle

$$(x,z) \to \left(e^{i\theta}x, e^{-i\theta}z\right),$$

instead of the usual mirror symmetry which uses three circles. As discussed in [32] this gives rise to a type-IIA description involving NS 5-brane wrapping the Riemann surface Σ given by $F(u,v) = 0$ and filling R^4. Without the extra D5 brane, this was used in [26] to derive the Seiberg-Witten vacuum geometry of $N = 2$ systems, where Σ was identified with the Seiberg-Witten Riemann surface. The only additional ingredient here is the fact that we also have an extra D5 brane, filling the z-plane. Since we are dualizing the circle which corresponds to phase rotations in the z-plane, the dual brane will have one less dimension, i.e., it will be a non-compact D4 brane, ending on the NS 5-brane in the type-IIA language. This gives us our third type-IIA description. If we now lift this up to M-theory, this gives a single M5 brane. In this formulation the Riemann surface that we obtain is related to the brane construction of MQCD [33, 34, 35], generalizing the construction of N=2 theories in [36], but the geometry is slightly different here.[5]

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